Optimal (r, nQ, T) Inventory Control under Stationary Demand

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Abstract. We consider the control of a single-echelon inventory system under the (r, nQ, T) ordering policy. Demand follows a stationary stochastic process and, when unsatisfied, is backordered. Under a standard cost structure, our aim is the minimization of the total average cost. In contrast to previous research, all policy variables (i.e. reorder level r, batch size Q and review interval T) are simultaneously optimized. While total average cost is not convex, two new convex bounds together with a Newsboy characterization of the optimal solution lead to an exact algorithm with guaranteed convergence to the global optimum. Computational results demonstrate that the inclusion of the review interval as a decision variable in the optimization problem can offer serious cost savings.

Keywords: Supply chain; Newsboy; Inventory; Optimal; Stochastic demand.

1. Introduction

The practical value of periodic review inventory control policies, where ordering decisions are taken in regular time intervals, is well established (e.g. Silver et al. 1998). Although theoretically inferior from their continuous review counterparts in terms of average cost performance (e.g. Veinott 1966 and Lee and Nahmias 1993), periodic policies offer practical advantages. By not imposing continuous monitoring of inventory status, they can be easily implemented in real production environments. By allowing for the routine overshoot of the reorder point, respective models can accommodate lumpy demands without loss of modeling accuracy. Hence, it is not surprising that the MRP logic, designed to deal with such demand processes, effectively implements standard periodic review ordering policies (e.g. Anderson and Lagodimos 1989 and Axsater and Rosling 1994).

Focusing on an established periodic review policy, the (r,nQ,T) policy, in this paper we study the optimal policy variables determination for a single-echelon inventory installation and propose an algorithm that guarantees cost-optimal control. While the system operating assumptions (i.e. stationary demand, full backordering and cost structure) are common in the field, there is an important differentiation from previous research. Earlier work has considered the (r, nQ) policy, a reduced version of the (r,nQ,T) policy, which assumes a fixed review interval T and where only two policy variables need be determined (the reorder level r and the batch size Q). In this paper all three policy variables (r,Q,T) of the original policy are simultaneously optimized, leading to solutions offering serious cost savings over those of the reduced problem.

We start with a review of key previous findings. The (r,nQ,T) policy was originally proposed by Morse (1959) as an adaptation of the (R,T) policy (also known as base-stock) to cope with quantized orders. These occur when supplies are constrained to be multiples of some basic batch-size Q, usually reflecting some physical limitation of the supply process (e.g. pallet-load, container load etc.). Considering the form of the policy, Veinott (1965) demonstrated that the (s,S) policy is optimal for externally fixed batch-size and zero ordering cost. Otherwise, the policy is clearly inferior compared to the more general policy, which is the known optimal policy for unrestricted supplies (e.g. Veinott 1966 and Federgruen and Zipkin 1985). This was further supported by the numerical results in Wagner et al. (1965) and Veinott and Wagner (1965) but who also suggested respective cost difference not to be large when comparing both policies at their optimal setting.

For independent demands, Hadley and Whitin (1961), building on the results by Morse, used Markov steady-state analysis to characterize the distribution of the inventory position of the (r,nQ,T) policy as being uniform $U(r,\ r+Q)$ irrespective of the demand distribution. As recently proposed by Li and Sridharan (2008), this distribution remains unchanged even for serially correlated demands. Using a general cost structure (the one also used here), Hadley and Whitin (1963) modelled long-run average cost of the (r,nQ,T) policy for Normal and Poisson demands. They also studied the cost function in terms of the policy variables $(r,\ Q,T)$. No analytical properties for the total cost were determined, so concluded that optimal control can only rely on exhaustive search approaches. Numerical comparisons with the (r,nQ,T) policy showed only marginal cost-differences as well as a tendency for the policy to often degenerate to $(R,\ T)$ at its optimal setting.

Considering the $(r,\ nQ)$ policy, Zheng and Chen (1992) proposed an algorithm to compute the ordering parameters (r and Q) that minimize long-run average cost for discrete uncorrelated demands. Under the cost structure in Hadley and Whitin but omitting review costs (since the policy assumes a fixed T), they proved average cost convexity in the reorder-level r together with a Newsboy-styled condition at the optimum. While the cost behaviour in Q was found erratic, Zheng and Chen proposed a convex bound, effectively identical to the average cost function in continuous review $(r,\ Q)$ policies (e.g. Zipkin 2000). Implementing the algorithm, they numerically compared the $(r,\ nQ)$ and (s,S) policies and only found small cost differences at the respective optima.

These papers, as well as a more recent by Larsen and Kiesmuller (2007) that models average cost for the $(r,\ nQ)$ policy under Erlang demand, all considered the review interval T as fixed, thus not entering the optimal control problem. The only previous study that (to our knowledge) considers T as a control variable is that by Rao (2003). Focusing on the $(R,\ T)$ policy under uncorrelated stationary demands, he showed that this can be analyzed as a limiting case of the continuous review $(r,\ Q)$ policy. This allowed him to show that the total average cost for the $(R,\ T)$ policy is jointly convex in both the order-up-to level R and the review interval T. Therefore any convex optimization algorithm can be used for optimal system control.

In this paper considering the (r, nQ, T) policy in its unrestricted form we propose an algorithm that guarantees convergence to the global optimum in all three policy variables.

The rest of the paper is organised as follows: In section 2 we present the notation and assumptions used through the paper. In section 3 the (r, nQ, T) policy's total average cost is modelled. In section 4 bounds for the total average cost are proposed and some properties of these bounds are obtained. An algorithm for the determination of the optimal policy's parameters is presented in section 5. In section 6 we give examples for the determination of the optimal controls assuming Normal distributed demand, under different cost settings. Finally some conclusions are given in section 7.

2. Notation and Assumptions

In this section we introduce the notation used together with the assumptions underlying the operation of the inventory system.

2.1 Notation

μ	The demand rate.
$D(t)$	Random variable denoting cumulative demand through time t, i.e demand in the interval [0, t] .
h, p	Inventory holding and backordering cost per unit per unit time respectively.
Ko	Fixed ordering cost (per ordering decision).
Kr	Fixed review cost (per review).
L	Replenishment lead time.
R	Upper starting inventory position limit (just after a review).
Q	Basic batch size (just after a review).
$I(R, Q, t)$	Net inventory position at time t.
P_o	Probability of ordering at any review period.
T	Length of review interval.
α	The α-service measure (non-stock-out probability).
x^+, x^-	The functions $x^+ = \max\{0, x\}$ $x^- = \max\{0 - x\}$.
$C(R, Q, T)$	Total average cost (per review interval).

2.2 Operating Assumptions

We study a single-item, single-echelon inventory installation controlled by *(r,nQ,T)* policy. Inventory status is reviewed every a time interval of length *T*. In the event an order is placed after a review, the order quantity is available after the elapse of a, deterministic and known, replenishment lead time *L*. Material leaves the installation in response to specific customer demands. Demand not satisfied from stock is backordered. The demand process is stochastically non-decreasing in t with mean $t\mu$, density $f(x,t)$ and cumulative distribution function $F(x, t)$. As is common in modelling cost we assume that cumulative demand has stationary and independent increments (see Serfozo and Stidham (1978)). These assumptions hold if the demand is modelled, for example, either as compound Poisson or Normal processes (see Rao (2003)). Note that the same assumptions for demand process are also used in Zipkin (1986) to obtain convexity properties for *(r, Q)* policy.

We also need to clarify the sequence of events within any review interval. (1) Replenishment orders placed respective lead time L earlier are received. (2) Inventory status is reviewed and a replenishment decision is taken. (3) Demand is realized. (4) Inventories and backorders are measured and relevant costs evaluated.

3. The *(r, nQ, T)* Inventory Policy

The *(r,nQ,T)* policy operates as follow: a) Inventory status is reviewed and ordering decisions taken at regular intervals of length *T*; b) If the inventory position is found to be below a reorder level r, a replenishment order is placed; c) Irrespectively of the taken decision the starting inventory position after any review epoch is given by $R\text{-}X(Q)$, and follows a uniform distribution U(r, R), where $X(Q)$ follows a uniform distribution $U(0,Q)$ and $R=r+Q$.

The size of an order (if it is placed) is nQ, where Q a predetermined batch size and n the smallest integer for which $R\text{-}X(Q) \geq r$. Note that for independent demands, it has been established (see Hadley and Whitin 1963) that $R\text{-}X(Q)$ follows a uniform distribution U(r, r+Q). As it was recently shown, the same distribution holds even for time-correlated demands (Li and Sridharan, 2008).

Remark 1. It is worthwhile to note that an alternative definition for *(R, T)* policy can be deduced setting Q= 0.

This remark allow a unified treatment for *(r, nQ, T)* and *(R, T)* policies.

3.1. The cost function under *(r,nQ,T)* policy

In this sub-section we model the average cost of the system and we obtain the expression for α-measure, which leads later to the Newsboy characterization of the optimal policy. We consider the general four element cost structure proposed in the seminal analysis by Hadley and Whitin (1963). In this context we assume linear holding and backordering penalty costs, as well as two fixed cost elements: ordering cost per actual replenishment order and review cost per review occasion. The review cost, K_r, incurred every T time units at each review and the ordering cost, K_o, incurred at the review instants where actual replenishment orders are released (so respective cost coefficient is multiplied by the ordering probability P_o). As discussed in Hadley and Whitin (1963), P_o represents the probability that demand between two consecutive reviews triggers an order at the second review so this clearly implies that $P_o = Pr(Q - X(Q) < D(T))$. Observe that P_o depends on Q and T but does not depend on R.

So, the inventory holding and backordering costs at time $t \in [0,T]$, have been pooled into the following function:

$$G(R,Q,t) = hE[(I(R,Q,t)^{+}] + pE[(I(R,Q,t)^{-}]$$

$$= hE[R - X(Q) - D(L+t))^{+}] + E[R - X(Q) - D(L+t))^{-}]$$

$$= h(R - \frac{Q}{2} - \mu(L+t)) + (h+p)E[(X(Q) + D(L+t) - R)^{+}]$$

$$= h(R - \frac{Q}{2} - \mu(L+t)) + \frac{h+p}{Q}\int_{R}^{+\infty}(y-R)\int_{0}^{Q} f(y-x,t)dxdy \qquad (1)$$

note that we use the facts that

$$E[(I(R,\ Q,\ t)] = E[(I(R,\ Q,\ t)^{+}] + E[(I(R,\ Q,\ t)^{-}]$$

and

$X(Q) + D(L+t)$ has density

$$\frac{1}{Q}\int_{0}^{Q} f(y-x,t)dx$$

Thus the average holding and backordering costs per unit time is

$$H(R,\ Q,T) = \frac{1}{T}\int_0^T G(R,Q,t)dt \tag{2}$$

and consequently the average total system cost per unit time can be expressed as:

$$C(R,Q,T) = \frac{K_r + K_o P_o}{T} + H(R,Q,T) \tag{3}$$

Now we obtain the expression for α-measure of service. Since the α-measure is defined as the non stock out probability we can easily see that

$$a(R,Q,T) = 1 - \frac{1}{T}\int_0^T \Pr(R - X(Q) - D(L+t)) \le 0 dt \tag{4}$$

4. Total Cost's Bounds and Properties

In this section we introduce two bounds to the cost function and present analytic properties that form the basis for the system optimal control.

Since the ordering probability always satisfies the relation, $0 \le P_o \le 1$, the following two bounds for $C(R,Q,T)$ directly prevail:

$$B_L(R,Q,T) = \frac{K_r}{T} + H(R,Q,T) \tag{5}$$

and

$$B_U(R,Q,T) = \frac{K_r + K_o}{T} + H(R,Q,T) \tag{6}$$

In the next lemma the optimal value of R is determined for given values of T and Q.

Lemma 1. Let $R(Q,T) = \arg\min_R C(R,Q,T)$ denotes the optimal R corresponding to fixed T and Q, then $R(Q,\ T)$ satisfies the condition $a(R(Q,T),Q,T) = \frac{p}{h+p}$, where $a(R(Q,T),Q,T)$ is the α-measure for this T and Q values.

Proof. From $\frac{dC(R,Q,T)}{dR} = 0$ we obtain the following Newsboy style equation

$$h - \frac{h+p}{T}\int_0^T \Pr(R - X(Q) - D(L+t)) \le 0 dt = 0$$

or

$$h - (h+p)(1 - a(R,Q,T)) = 0$$

and finally we get

$$a(R(Q,T),Q,T) = \frac{p}{h+p} \tag{7}$$

Direct application of this result clearly reduces the problem state-space (by one variable), thus facilitating optimal control parameters.

Notice that the optimal R for fixed T and Q is the same for the average cost per unit time, $C(R, Q, T)$, and for its bounds, $B_L(R, Q, T)$ and $B_U(R, Q, T)$ so the next lemma follows easily:

Lemma 2. $R(Q,T) = \arg\min_R C(R,Q,T) = \arg\min_R B_L(R,Q,T) = \arg\min_R B_U(R,Q,T)$

Lemma 3. For every given T let $R(Q;T) = \arg\min_R C(R, Q,T)$ denotes the optimal R corresponding to Q, then $B_L(R(Q;T), Q, T)$ and $B_U(R(Q;T), Q, T)$ are increasing and convex functions in Q.

Proof. Zheng (1992) in Lemma 4 prove that the function $G(R(Q;T), Q, t)$ is an increasing and convex function of Q so the same holds for the functions $B_L(R(Q;T), Q, T)$ and $B_U(R(Q;T), Q, T)$ (see also figure 1).

It is worthwhile to note that $B_L(R, Q, T)$ and $B_U(R, Q, T)$ represent the cost for systems, which operate under a (r, nQ, T) policy but it forces to order at each review epoch (under different fixed costs). In such circumstance it is know that these systems require $Q=0$ (Veinott 1966) and consequently it is optimal for these systems to operate under (R, T) policies.

Lemma 4. Let $R(0,T) = \arg\min_R C(R, 0,T)$ denotes the optimal R corresponding to T for Q=0, if $D(t)$ is stochastically increasing linearly then $B_L(R(0,T), 0, T)$ and $B_U(R(0,T), 0, T)$ are convex function in T.

Proof. From Rao (2003) Theorem 6, $B_L(R, 0, T)$ and $B_U(R, 0, T)$ are jointly convex in R and T so $B_L(R(0,T), 0, T)$ and $B_U(R(0,T), 0, T)$ are convex function in T (see also figure 2).

It is interesting to observe that in the above both $B_L(R(0,T), 0, T)$ and $B_U(R(0,T), 0, T)$ represent the optimal cost of (R,T) policies (by definition obtained from the (r,nQ,T) policy with $Q=0$) for given T. Therefore, the optimal (r,nQ,T) policy is bounded above and below by specific (R,T) policies derived from the bounds in (5) and (6).

The next result follows naturally.

Lemma 5. $\lim_{T\to\infty}[B_U(R,\ Q,\ T)-B_L(R,\ Q,\ T)]=0$

5. Optimal Control

In this section we present an algorithm for the determination of the optimal values for $(R,\ Q,\ T)$. This algorithm converges to the optimal in a finite number of steps for any given accuracy.

`Algorithm` $R^*,Q^*,T^* = Min < R,nQ,T > Cost(K_r,K_o,L,\mu,\sigma,h,p)$

`inputs:` review cost, ordering cost, lead-time, demand distribution mean and variance, holding and backorders cost coefficients

`outputs:` control parameters for order-up-to level R, order quantum Q (in multiples of a quantum order Q_q, review period interval T (in multiples of a time quantum T_q)

1. set $C^* = +\infty$
2. for $T=T_q$, $2T_q$... do
   a. for $Q=Q_q$, $2Q_q$... do
      i. let $R' = \arg\min_R C(R,Q,T)$;
      ii. let $C^{Q,T} = C(R',Q,T)$, $B_L^{Q,T} = \min_R B_L(R,Q,T)$;
      iii. if $C^{Q,T} < C^*$
         1. set $C^* = C^{Q,T}$, $R^* = R', Q^* = Q, T^* = T$;
      iv. end if;
      v. if $B_L^{Q,T} > C^*$ break;
   b. end for;
   c. let $B_T^* = \min_{R,Q} B_L(R,Q,T)$;
   d. if $B_T^* > C^*$ break;
3. end for
4. return (R^*,Q^*,T^*);

End.

The algorithm is guaranteed to terminate as the lower bound of the function goes to infinity as $T \to +\infty$ and $B_L(R(Q;T),\ Q,\ T)$ is increasing and convex functions in Q and also

$\lim_{Q\to\infty} B_L(R_{Q,T},Q,T) = +\infty$. Therefore, the conditions in steps 2.a.v as well as 2.d will eventually be met and the algorithm will terminate. The conditions are also sufficient:

1. For the case of step 2.a.v, there is no point in searching for any higher Q as it is guaranteed that the cost function, being greater than the lower bound will always be greater than our current incumbent value, as all other values in the range $[T,\ldots]\text{x}[Q,\ldots]$ will yield higher costs (the lower bound is now increasing in Q).
2. For the case of step 2.d it is obvious that at the value of T for which the condition is met, the sequence B_T^* is increasing (otherwise it would have been impossible to have found a cost value less than the lower bound) and thus, from now on the sequence $C_T^* = \min_{(R,Q)} C(R,Q,T)$ will always be above the current C^* which becomes the global optimum.

The previous results summarized in the following:

Proposition. The proposed algorithm converges to the optimal in a finite number of steps.

6. Numerical Results

We have applied the proposed procedure to determine the optimal controls assuming Normal distributed demand with $E(D(t)) = t\mu$ and $Var(D(t)) = t\sigma^2$, with μ=10, σ=3, L=5 for a number of different cost coefficients (the formula for total average cost assuming Normal distributed demand is derived in the Appendix). In addition note (see Rao, 2003) that all feasible reorder intervals must be at least $t_{\min}$ and demand rate μ is sufficiently larger than σ ($\mu > 3\sigma$) so that $\mu t >> \sigma\sqrt{t}$ for al $t \geq t_{\min}$ and consequently the probability of negative demand is negligibly small for $T \geq t_{\min}$. The results are shown in Table 1 below. The first four columns in Table 1 determine the cost coefficients of the problem. The columns entitled Ropt and Topt under the heading "(R,T) Policy Optimization" are the optimal controls of the (R,T) policy applied to the problem, and RTcost is the optimal cost of the (R,T) policy. The columns r^*, Q^* and T^* under the heading "(R,nQ,T) Policy Optimization" denote the optimal controls for the policy (R,nQ,T) when all three parameters are allowed to vary, and the column denoted $C^*(R^*,Q^*,T^*)$ denotes the optimal value of the cost function of the (R,nQ,T)

policy. Finally, the last 3 columns determine the optimal controls R^* and Q^* together with the value for the optimal (R,nQ,T) policy when the time parameter (the length of the period) is arbitrarily set to 1 and not allowed to vary.

The rows in bold are the cases where the (R,nQ,T) policy is strictly better than the simpler (R,T) policy. As can be seen, the differences of the policies in terms of the optimal cost are relatively small in all cases; in most cases, the optimal (R,nQ,T) policy reduces to the (R,T) policy. Nevertheless, notice the important role the T parameter (length of period) can play in the optimal cost determination. For example, for the case K_r=250, K_o=1, h=10, p=1, the optimal (R,nQ,T) policy is more than 270% better than the optimal policy determined by fixing the parameter T=1 !.

7. Conclusions

In this paper we developed an algorithm for computing optimal *(r, nQ, T)*. This algorithm is constructed incorporating results for *(r, Q)* and *(R, T)* policies. To the best of our knowledge only results for *(r, nQ)* policy, a special case of *(r, nQ, T)* policy with T=1, have been presented by Zheng and Chen (1992) and recently by Larsen Kiesmuller (2007). The computational findings presented in the previous section indicate serious cost savings when the parameter T is a decision variable. In addition a close relation between the *(r, nQ, T)* and *(R, T)* is concluded. From $B_U(R,Q,T)$ the optimal cost of the *(R, T)* is an upper bound for the optimal cost of *(r, nQ, T)*. While the numerical results show that in many cases the two costs coincide.

Table 1. Optimal Control of (R,T), (R,nQ,T) & $(R,nQ,T$=1) Policies

L=5, μ=10, σ=3				(R,T) Policy Optimization			(r,nQ,T) Policy Optimization				(r,nQ,T=1) Policy		
Kr	Ko	h	p	Ropt	Topt	RTcost	r*=R*-Q*	Q*	T*	C*(R*,Q*,T*)	r*=R1*-Q1*	Q1*	C*(R*,Q*,T=1)
1	1	1	1	56,1	1,24	7,98	55,9	0	1,2	7,98	54,93	0	8,07
1	1	1	10	64,43	0,89	15,79	64,53	0	0,9	15,79	65,22	0	15,83
1	1	1	100	70,93	0,76	22,32	71,29	0	0,8	22,33	72,95	0	22,59
1	1	10	1	44,79	0,97	15,51	44,89	0	1	15,51	44,89	0	15,51
1	1	10	10	52,57	0,52	60,04	52,48	0	0,5	60,02	54,93	0	62,73
1	1	100	1	37,3	0,87	21,78	37,36	0	0,9	21,78	37,61	0	21,83
1	50	1	1	73,12	4,71	24,01	73,07	0	4,7	24,01	31,5	47	24,62
1	**50**	**1**	**10**	**85,28**	**3,51**	**37,62**	**53,17**	**37**	**0,8**	**37,26**	54,36	37	37,34
1	**50**	**1**	**100**	**93,3**	**3,17**	**46,96**	**62,09**	**35**	**0,6**	**45,56**	65,02	35	46,02
1	50	10	1	50,78	3,67	35,96	50,82	0	3,7	35,96	18,63	37	37,22
1	**50**	**10**	**10**	**58,71**	**1,78**	**100,21**	**43,49**	**18**	**0,5**	**100,07**	45,46	19	102,69
1	**50**	**10**	**100**	**67,26**	**1,28**	**186,37**	**55,18**	**14**	**0,3**	**182,28**	65,22	0	189,33
1	50	100	1	41,19	3,49	43,69	41,21	0	3,5	43,69	10,28	35	45,54
1	50	100	10	46,08	1,39	181,22	46,09	0	1,4	181,22	44,89	0	186,11
1	250	1	1	100,1	10,11	51	100,05	0	10,1	51	4,5	101	51,58
1	**250**	**1**	**10**	**120,49**	**7,54**	**73,77**	**48,35**	**77**	**0,9**	**72,2**	48,89	77	72,2
1	**250**	**1**	**100**	**130,55**	**6,94**	**86,54**	**60,01**	**74**	**0,7**	**82,15**	62,1	74	82,48
1	250	10	1	55,93	7,7	71,03	55,93	0	7,7	71,03	0	61	74,3
1	**250**	**10**	**10**	**66,75**	**3,42**	**176,45**	**35**	**35**	**0,5**	**175,08**	37,49	35	176,38
1	250	100	1	43,8	7,41	80,3	43,8	0	7,4	80,3	0,01	48	89,25
1	250	100	10	49,06	2,69	278,28	49,08	0	2,7	278,29	26,76	27	285,24
250	1	1	1	100,1	10,11	51	100,05	0	10,1	51	54,93	0	257,07
250	1	1	10	120,49	7,54	73,77	120,11	0	7,5	73,77	65,22	0	264,83
250	1	1	100	130,55	6,94	86,54	130,15	0	6,9	86,54	72,95	0	271,59
250	1	10	1	55,93	7,7	71,03	55,93	0	7,7	71,03	44,89	0	264,51
250	1	10	10	66,75	3,42	176,45	66,62	0	3,4	176,45	54,93	0	311,73
250	1	10	100	77,1	2,53	290,51	76,82	0	2,5	290,53	65,22	0	389,35
250	1	100	1	43,8	7,41	80,3	43,8	0	7,4	80,3	37,61	0	270,83
250	1	100	10	49,06	2,69	278,28	49,08	0	2,7	278,29	44,89	0	386,13
250	1	100	100	56,64	1,35	837,18	56,87	0	1,4	837,48	54,93	0	858,33
250	50	1	1	104,36	10,99	55,63	104,55	0	11	55,63	31,5	47	273,62
250	50	1	10	126,59	8,23	79,98	126,32	0	8,2	79,98	54,36	37	286,34
250	50	1	100	136,98	7,59	93,29	137,12	0	7,6	93,29	65,02	35	295,02
250	50	10	1	56,67	8,38	77,13	56,69	0	8,4	77,13	18,63	37	286,22
250	50	10	10	68,12	3,7	190,2	68,1	0	3,7	190,2	45,46	19	351,69
250	50	10	100	78,86	2,75	309,07	78,47	0	2,7	309,1	65,22	0	438,33
250	50	100	1	44,13	8,08	86,63	44,14	0	8,1	86,63	10,28	35	294,54
250	50	100	10	49,47	2,9	295,81	49,46	0	2,9	295,81	44,89	0	435,11
250	50	100	100	57,11	1,45	872,17	56,87	0	1,4	872,47	54,93	0	907,3
250	250	1	1	120,58	14,21	71,48	120,55	0	14,2	71,48	4,5	101	300,58
250	250	1	10	147,56	10,58	101,25	147,73	0	10,6	101,25	48,89	77	321,2
250	250	1	100	159,04	9,8	116,3	159,05	0	9,8	116,3	62,1	74	331,48
250	250	10	1	59,05	10,71	98,08	59,04	0	10,7	98,08	0	61	323,3
250	250	10	10	72,9	4,67	237,97	73,07	0	4,7	237,98	37,49	35	425,38
250	250	10	100	85,01	3,48	373,3	85,19	0	3,5	373,31	56,25	27	535,87
250	250	100	1	45,1	10,34	108,34	45,08	0	10,3	108,34	0,01	48	338,25
250	250	100	10	50,72	3,64	356,88	50,66	0	3,6	356,9	26,76	27	534,24
250	250	100	100	58,64	1,76	996,49	58,81	0	1,8	996,64	54,93	0	1107,22

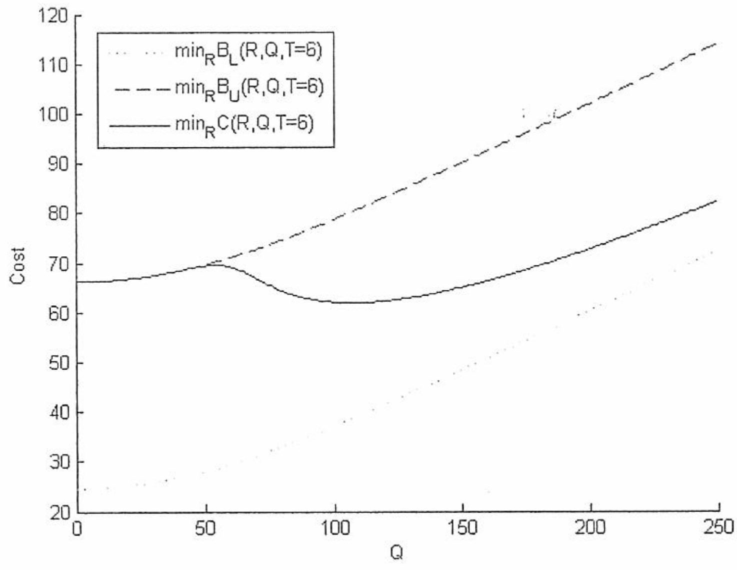

Figure 1: Plot of the cost function and its bounds as a function of the base order quantity Q at their minimum over R, for K_r=50, K_o=250, L=5,μ=10,σ=3,h=1,p=1,T=6. Both bounds are convex increasing. Notice however that the function $C(Q)$ has two local minimums (the first at Q=0). The upper bound coincides with the cost function for Q<50.

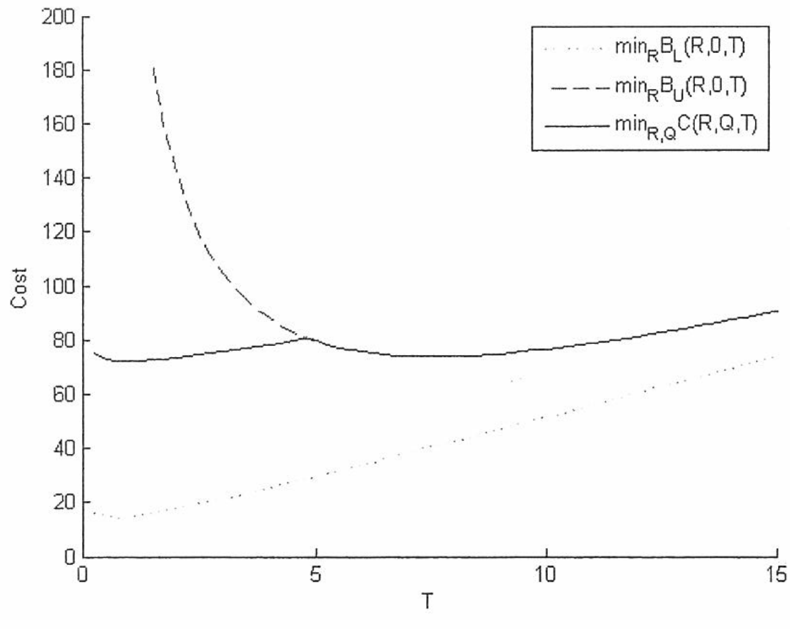

Figure 2: Plot of the cost function and its bounds as a function of the period length T at their minimum over R and Q, for K_r=1, K_o=250, L=5, μ=10, σ=3, h=1, p=10. Both bounds are convex but not increasing in T. The function $C(T)$ has again two local minimums (the first is the global minimum). Also notice that the upper bound coincides with the actual cost function for T>5.

Appendix: Closed-form expressions for Normal demand

Assuming Normal distributed demand, we now obtain the total average cost and a-service measure. Since we assume uncorrelated Normal distributed demand, the demand over t consecutive periods is also Normal with $E(D(t)) = t\mu$ and $Var(D(t)) = t\sigma^2$. In the following, we make use of the standardizing ratios:

$$Z_t = \frac{R - Q - t\mu}{\sigma\sqrt{t}}\left[= \frac{r - t\mu}{\sigma\sqrt{t}}\right],\ R_t = \frac{Q}{\sigma\sqrt{t}},\ M_t = \frac{t\mu}{\sigma\sqrt{t}} \tag{8}$$

We start by modeling average total cost *C(R, Q, T)* in (3). So firstly, we need to model the ordering probability P_o. By standardizing the variable *D(T)* this can be expressed as:

$$P_o = \Pr(u > w) = 1 - \Pr(u \le w) = 1 - \frac{1}{R_T}\int_{-M_T}^{R_T - M_T} \Phi(x)dx$$

where

$u \sim N(0,1)$ and $w \sim U(-M_T, R_T - M_T)$ and $\Phi(.)$ is the cumulative distribution function for the standard Normal.

We can directly evaluate the integral above (using integration by parts) and obtain the average fixed costs, say Θ(Q,T):

$$\Theta(Q,T) = \frac{K_r + K_o}{T} - \frac{K_o}{TR_T}(\varphi(R_T - M_T) + (R_T - M_T)\Phi(R_T - M_T) - \varphi(M_T) + M_T\Phi(-M_T) \tag{9}$$

where $\varphi(.)$ is the density function for the standard Normal. We consider now average holding and backordering cost, $H(R,\ Q,\ T)$. Since $E(R - X(Q)) = R - \frac{Q}{2} = r + \frac{Q}{2}$ we only need to determine $E(I(R,Q,t)^-)$. By standardizing the Normal variable *D(L+t)*, this can be expressed in terms of the variables $u \sim N(0,1)$ and $w \sim U(0, R_{L+t})$

So

$$E(I(I,R,t)^-) = \frac{\sigma\sqrt{L+t}}{R_{L+t}}\int_{Z_{L+t}}^{Z_{L+t}+R_{L+t}}\int_x^{\infty}(y-x)\phi(y)dydx = \frac{\sigma\sqrt{L+t}}{R_{L+\tau}}\int_{Z_{L+t}}^{Z_{L+t}+R_{L+t}}[\phi(x) - x + x\Phi(x)]dx. \tag{10}$$

Using again integration by parts (twice), after some algebra a closed-form expression for $E(I(R,Q,t)^-)$ is obtained. So, we finally get $H(R,\ Q,\ T)$ as:

$$H(R,Q,T)=h[R-\frac{Q}{2}-\mu(L+\frac{T+1}{2})]+\frac{h+p}{2T}\int_0^T\frac{\sigma\sqrt{L+t}}{R_{L+t}}\{[(Z_{L+t}+R_{L+t})^2+1]\Phi(Z_{L+t}+R_{L+t}) \quad (11)$$
$$+(Z_{L+t}+R_{L+t})\phi(Z_{L+t}+R_{L+t})-(Z_{L+t}^2+1)\Phi(Z_{L+t})-Z_{L+t}\phi(Z_{L+t})-R_{L+\tau}(2Z_{L+t}+R_{L+t})\}dt.$$

Thus, a closed-form expression for average total cost model under Normal distributed demand is now fully determined as the sum of (9) and (11).

In order to apply the Newsboy-styled condition we also need to model $a(R,Q,T)$. But, this is nearly identical to the ordering probability P_o, so it can be modeled analogously. Using identical steps, we finally obtain:

$$a(R,Q,T)=\int_0^T\frac{1}{TR_{L+t}}\{\phi(Z_{L+t}+R_{L+t})+(Z_{L+t}+R_{L+t})\Phi(Z_{L+t}+R_{L+t})-\phi(Z_{L+t})-Z_{L+t}\Phi(Z_{L+t})\}dt \quad (12)$$

which, determines a closed-form expression for the α-measure under the conditions considered.

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